

Constraints on light quark masses from the heavy meson spectrum

Markus A. Luty

*Center for Theoretical Physics
Massachusetts Institute of Technology
Cambridge, MA 02139*

Raman Sundrum

*Lyman Laboratory of Physics
Harvard University
Cambridge, MA 02138*

(February 1995)

Abstract

We use the observed $SU(3)$ breaking in the mass spectrum of mesons containing a single heavy quark to place restrictions on the light quark current masses. A crucial ingredient in this analysis is our recent first-principles calculation of the electromagnetic contribution to the isospin-violating mass splittings. We also pay special attention to the role of higher-order corrections in chiral perturbation theory. We find that large corrections are necessary for the heavy meson data to be consistent with $m_u = 0$. We use the observed $SU(3)$ breaking in the mass spectrum of mesons containing a single heavy quark to place restrictions on the light quark current masses. A crucial ingredient in this analysis is our recent first-principles calculation of the electromagnetic contribution to the isospin-violating mass splittings. We also pay special attention to the role of higher-order corrections in chiral perturbation theory. We find that large corrections are necessary for the heavy meson data to be consistent with $m_u = 0$.

I. INTRODUCTION

The light quark current masses m_u , m_d , and m_s are among the fundamental parameters in the standard model of particle interactions, but an accurate and reliable determination of these parameters remains elusive. The reason is that these masses are small compared to the mass scale associated with confinement, $\Lambda \sim 1$ GeV, so that the light quarks are tightly bound inside hadrons and their mass cannot be measured directly. However, we can expand around the chiral limit $m_{u,d,s} \rightarrow 0$ to obtain information about current mass *ratios* using chiral perturbation theory [1–3]. The idea is that in the chiral limit, there is a $SU(3)_L \times SU(3)_R$ chiral symmetry that is spontaneously broken to $SU(3)_{L+R}$ at the scale Λ . In this limit, the theory contains 8 Nambu–Goldstone bosons, which are identified with the light pseudoscalar meson octet (π , K , η). The low-energy interactions of these states can be parameterized by an effective lagrangian with a few unknown parameters [4]. In addition, the chiral symmetry is explicitly broken by the light quark masses and by electromagnetism. This breaking can be treated perturbatively in the quark masses and the electromagnetic coupling, and selection rules associated with the chiral symmetry again tightly constrain the form of these perturbations.

The main difficulty in this approach is that $m_s/\Lambda \sim 20\text{--}30\%$, so that higher-order effects can significantly change low-order results. This is the source of the interesting (and controversial) issue of whether higher-order corrections can be large enough to allow $m_u = 0$ [5–7], thus solving the the strong CP problem. This will be a large part of the focus of this paper.

In this paper, we apply the methods of heavy quark symmetry [8–10] and heavy hadron chiral perturbation theory [11] to the determination of the light quark masses from the mass spectrum of mesons containing a single heavy quark. These states have quantum numbers $P_q \sim Q\bar{q}$, where $Q = b, c$ and $q = u, d, s$. If we could ignore electromagnetic effects and higher-order corrections in the quark masses, we would obtain

$$R \equiv \frac{m_s - \hat{m}}{m_d - m_u} \stackrel{?}{=} \frac{P_s - \hat{P}}{P_d - P_u} \simeq \begin{cases} 20 & \text{for } P = D \\ 280 & \text{for } P = B \end{cases} \quad (1.1)$$

where $\hat{m} \equiv \frac{1}{2}(m_u + m_d)$, *etc.* (We use the names of the heavy-meson states to denote their masses.) The enormous discrepancies between these numbers clearly cannot be accounted for by the higher-order quark mass corrections. The reason for these discrepancies is simply that the electromagnetic contribution to $P_d - P_u$ is numerically comparable to the contribution from the light quark masses. This underlines the importance of determining the electromagnetic contribution to the mass differences.

We will carry out an improved analysis of R making use of the heavy-meson electromagnetic mass differences computed in ref. [12]. (For earlier related work, see ref. [13].) Our analysis will also include important non-analytic corrections in the quark masses [14,15] and will pay careful attention to the role of higher-order corrections in chiral perturbation theory. We will briefly describe the elements of the calculation of the electromagnetic mass differences in the next section. The following section contains our analysis of R , and the final section gives our conclusions.

II. COMPUTATION OF ELECTROMAGNETIC MASS DIFFERENCES

In ref. [12], we computed the electromagnetic mass differences of heavy mesons in terms of measurable data using techniques similar to those used in the classic calculation of the $\pi^+-\pi^0$ mass difference [16]. The basic idea in both calculations is to use dispersion-theoretic arguments together with the ultraviolet properties of QCD to relate the electromagnetic mass difference to measured strong-interaction matrix elements. We review the main features of our calculation here in order to make this paper more self-contained.

We begin by writing

$$(P_d - P_u)^{(\text{EM})} = \frac{ie^2}{2} \int \frac{d^4k}{(2\pi)^4} \frac{\Delta T(p, k)}{k^2 + i0+} \quad (2.1)$$

where $\Delta T \equiv T_d - T_u$ is a difference of the (traced) Compton amplitudes

$$T_q(p, k) \equiv i \int d^4x e^{ik \cdot x} \langle P_q(p) | J^\mu(0) J_\mu(x) | P_q(p) \rangle, \quad (2.2)$$

where J^μ is the electromagnetic current. It can be shown rigorously that this integral converges [17,12].

We consider ΔT in the large- N limit, where N is the number of QCD colors. In this limit, we can express ΔT as a sum of tree graphs in a theory of (infinitely many) mesons with interactions that are at most polynomial in momenta [18]. (This can be viewed as a rigorous version of the “resonance dominance” assumption made in refs. [16].) The vertices of the meson graphs that determine ΔT are directly related to physical processes. For example, the graphs in fig. 1b determine the heavy-meson form factors, while the last two graphs in fig. 1a are related to meson scattering amplitudes. Thus, the representation of ΔT in the meson theory can be viewed as a double dispersion relation in k and p .

We then impose constraints on the couplings appearing in this sum by demanding that the matrix elements have the infrared behavior required by electromagnetic gauge invariance, and the ultraviolet behavior demanded by QCD. These constraints can be expressed as sum rules relating the couplings appearing in the sum. (Two of these sum rules are close analogs of the Weinberg sum rules that appear in the calculation of the $\pi^+-\pi^0$ mass difference [19].)

Even after imposing the sum rules, the electromagnetic mass difference is given by an infinite (convergent) sum over intermediate states. We then derive an unsubtracted fixed- \vec{k}^2 dispersion relation for ΔT that shows that the contributions of intermediate states with large mass are suppressed by inverse powers of their mass. It is therefore sensible to assume that the sum is dominated by the lowest-lying intermediate states. That is, we keep the minimum number of intermediate states to saturate the known asymptotic behavior of the form factors and other matrix elements that appear. This turns out to be the P and P^* heavy-meson states (which are degenerate in the limit $m_Q \rightarrow \infty$) and the vector mesons ρ , ω , ρ' and ω' that are responsible for “softening” the form factors. When the sum rules are imposed on this restricted set of intermediate states, there are sufficiently few free parameters that we are able to obtain numerical estimates for the electromagnetic mass differences. The results are

$$(B^+ - B^0)^{(\text{EM})} \simeq \left[+1.7 - 0.13 \left(\frac{\beta}{1 \text{ GeV}^{-1}} \right) - 0.03 \left(\frac{\beta}{1 \text{ GeV}^{-1}} \right)^2 \right] \text{ MeV} + O(1/m_b^2), \quad (2.3)$$

$$\begin{aligned} \frac{1}{4} [(D^+ - D^0) + 3(D^{*+} - D^{*0})]^{(\text{EM})} &\simeq \left[2.5 + 0.012 \left(\frac{\beta}{1 \text{ GeV}^{-1}} \right)^2 \right. \\ &\quad \left. + 0.011 \left(\frac{\beta}{1 \text{ GeV}^{-1}} \right)^2 \left(\frac{\beta - \beta'}{0.3\beta} \right) \right] \text{ MeV} \\ &\quad + O(1/m_c^2), \end{aligned} \quad (2.4)$$

$$\begin{aligned} [(D^{*+} - D^{*0}) - (D^+ - D^0)]^{(\text{EM})} &\simeq \left[0.16 + 0.99 \left(\frac{\beta}{1 \text{ GeV}^{-1}} \right) \right. \\ &\quad \left. + 0.015 \left(\frac{\beta}{1 \text{ GeV}^{-1}} \right)^2 \left(\frac{\beta - \beta'}{0.3\beta} \right) \right] \text{ MeV} \\ &\quad + O(1/m_c^2). \end{aligned} \quad (2.5)$$

Here β (β') measure $P^*P\gamma$ ($P^*P^*\gamma$) couplings. Heavy-quark symmetry gives $\beta = \beta' + O(1/m_Q)$, and experimental constraints from D^* decays give $0 < -\beta \leq 5 \text{ GeV}^{-1}$ [20,21].¹ Since the isospin splittings in the P and P^* systems are equal in the heavy-quark limit, we cannot use them to get independent constraints on the quark masses. For the D system, we have chosen to work in terms of the “spin-independent” combination $D + 3D^*$ [15], for which the the largest β -dependent term in the electromagnetic mass difference cancels. We will use the D hyperfine splitting $D^* - D$ to give a determination of β below.

There are a number of simultaneous approximations made in this computation: the heavy-quark expansion (including $1/m_Q$ terms), the large- N limit, and the truncation of intermediate states. The heavy-quark expansion appears to be under good control in our calculation, in the sense that the $1/m_Q$ corrections are small compared to the leading terms. The large- N limit and truncation of intermediate states are equivalent to approximations that can be made in the computation of the $\pi^+ - \pi^0$ mass difference, which then works to 30%. We conclude that it is reasonable to expect our estimates of the electromagnetic mass differences to be accurate to about 30%.

We now complete our determination of the electromagnetic mass differences by extracting a value for β from the D hyperfine splittings. To order m_q/m_Q in chiral perturbation theory, one finds [15]

$$\begin{aligned} (D^{*+} - D^{*0}) - (D^+ - D^0) &= [(D^{*+} - D^{*0}) - (D^+ - D^0)]^{(\text{EM})} \\ &\quad + \frac{1}{R} [(D_s^* - \hat{D}^*) - (D_s - \hat{D})]. \end{aligned} \quad (2.6)$$

Experimentally, $(D_s^* - \hat{D}^*) - (D_s - \hat{D}) = 0.5 \pm 2.5 \text{ MeV}$, and $R \geq 20$. Thus, the second term on the right-hand side is negligible, and the isospin hyperfine splitting is dominated by the electromagnetic contribution. Using eq. (2.5), we obtain

$$\beta \simeq -1.6 \text{ GeV}^{-1} \quad (2.7)$$

¹Our convention for β differs by a sign from that of ref. [20].

for reasonable values of β' . This is consistent with the results of refs. [20,21]. For comparison the constituent quark model prediction is $\beta \simeq -3 \text{ GeV}^{-1}$. It should be emphasized that our determination has errors $O(1/m_c)$, and there is a further $O(1/m_c)$ error in using this value for the B system. Using eq. (2.7), we obtain

$$\frac{1}{4} [(D^+ - D^0) + 3(D^{*+} - D^{*0})]^{(\text{EM})} \simeq 2.5 \text{ MeV}, \quad (2.8)$$

$$(B^+ - B^0)^{(\text{EM})} \simeq 1.8 \text{ MeV}. \quad (2.9)$$

The electromagnetic mass differences are not very sensitive to β , so despite the uncertainties discussed above, we expect these values to be correct to about 30%.

III. CONSTRAINTS ON R

We now turn to the analysis of R defined in eq. (1.1). We pay special attention to the role of higher-order corrections in chiral perturbation theory, since it is known that these can have an important effect on the determination of the light quark mass ratios [5,7]. Keeping terms of order m_q , $m_q^{3/2}$, m_q/m_Q , m_q^2 , and the corresponding logs,² the mass differences of the heavy mesons can be written in the form

$$P_d - P_u = (P_d - P_u)^{(\text{EM})} + A_0(m_d - m_u) + \Delta_{d-u}^{(3/2)} + [A_1(\mu) + \Delta_1(\mu)] \frac{m_d - m_u}{m_Q} \\ + [A_2(\mu) + \Delta_2(\mu)] m_s(m_d - m_u) + O(m_{u,d}^2 \ln m_s) \quad (3.1)$$

$$P_s - \hat{P} = A_0 m_s + \Delta_s^{(3/2)} + [A_1(\mu) + \Delta_1(\mu)] \frac{m_s}{m_Q} \\ + [A_2(\mu) + \Delta_2(\mu)] m_s^2 + [A_3(\mu) + \Delta_3(\mu)] m_s^2 + O(m_s^{5/2}). \quad (3.2)$$

(For the D system, $P = \frac{1}{4}(D + 3D^*)$.) The meaning of these terms is as follows: $(P_d - P_u)^{(\text{EM})}$ is the electromagnetic contribution discussed in the previous sections. $A_0 m_q$ is the term linear in quark masses that gives rise to eq. (1.1). $\Delta^{(3/2)}$ are the $O(m_q^{3/2})$ non-analytic corrections, which can be expressed in terms of the light meson masses and couplings [14,15]:³

$$\Delta_{d-u}^{(3/2)} = -\frac{g^2}{16\pi f^2} (M_{K^0}^3 - M_{K^+}^3 + \frac{1}{2R} M_\eta^3), \quad (3.3)$$

$$\Delta_s^{(3/2)} = -\frac{g^2}{16\pi f^2} (M_K^3 + \frac{1}{2} M_\eta^3). \quad (3.4)$$

Here, $f \simeq 110 \text{ MeV}$ is the K decay constant and g measures the strength of the $PP\pi$ coupling [11]. The experimental limit on the D^* width gives the bound $g^2 \leq 0.5$ [11]. (To

²This corresponds to assuming that $m_s/\Lambda \sim \Lambda/m_Q$. This probably overestimates the $O(m_q/m_Q)$ corrections for the B system, but this is harmless because our results are identical if these corrections are omitted.

³Ref. [14] contains an error that was corrected in ref. [15].

the order we are working, the value of g is the same in the D and B system by heavy quark flavor symmetry.) However, the analysis of refs. [20,21] reveals that the magnitudes of g and β are correlated, so that $g^2 \simeq 0.15$ for the value of β in eq. (2.7). Their analysis also shows that $g^2 \geq 0.1$. We will conservatively assume $0.1 \leq g^2 \leq 0.3$, which corresponds to $-\beta \leq 3 \text{ GeV}^{-1}$.

The terms proportional to A_1 and $A_{2,3}$ in eqs. (3.1,3.2) arise from analytic counterterms in the effective lagrangian of order m_q/m_Q and m_q^2 , respectively. The terms proportional to $\Delta_{1,2,3} \sim \ln m_s$ are the chiral logs that renormalize them. The counterterms and the chiral logs each depend on a renormalization scale μ in such a way that the terms proportional to $A_j + \Delta_j$ are independent of μ . The fact that the chiral logs are proportional to the same function of quark masses as the corresponding counterterms is a consequence of the simple structure of the $SU(3)$ group theory for this system.

Perhaps surprisingly, the chiral logs in $\Delta_{2,3}$ cannot be computed in terms of known quantities. The reason is the presence of higher-order terms in the effective heavy-meson lagrangian such as (in the notation of ref. [11])

$$\delta\mathcal{L} = \frac{c}{\Lambda} \text{tr} \left[H^\dagger H (\xi^\dagger \partial^\mu \xi - \xi \partial^\mu \xi^\dagger) (\xi^\dagger \partial_\mu \xi - \xi \partial_\mu \xi^\dagger) \right], \quad (3.5)$$

that are not constrained by present data. These terms give rise to non-analytic contributions such as

$$\Delta_3 m_s^2 \sim \frac{c}{\Lambda} \frac{M_K^4}{16\pi^2 f^2} \ln \frac{M_K^2}{\mu^2} = O(m_s^2 \ln m_s). \quad (3.6)$$

(A similar fact was noted for baryons in ref. [22].) In principle, the log corrections are enhanced over the counterterm contributions by $\sim \ln M_K^2/\mu^2$ for $\mu \sim 1 \text{ GeV}$, but in practice the logs are not significantly larger than unity. We will therefore treat the terms proportional to $A_j + \Delta_j$ as unknown corrections.

We can solve for R from eqs. (3.1,3.2) to obtain

$$R = \frac{(P_s - \hat{P}) - \Delta_s^{(3/2)}(K) - \bar{A}_3}{(P_d - P_u) - (P_d - P_u)^{(\text{EM})} - \Delta_{d-u}^{(3/2)}(K)}. \quad (3.7)$$

where $\Delta^{(3/2)}(K)$ are the terms in the $O(m_q^{3/2})$ corrections which depending on the K masses (see eqs. (3.3,3.4)); $\bar{A}_3 \equiv [A_3(\mu) + \Delta_3(\mu)] m_s^2$ parameterizes the unknown $O(m_q^2)$ and $O(m_q^2 \ln m_q)$ deviations from the $O(m_q^{3/2})$ mass relations for both the D and B systems. Since such relations are expected to work to 20–30%, it is reasonable to assume that \bar{A}_3 should not be much larger than 30%. Ref. [5] advocates a different measure of the chiral corrections. They demand that the second-order corrections to individual masses be less than 30%, but allow this to be the result of cancellations between larger corrections. Following this criterion, we would allow \bar{A}_3 to be 60%, since this can cancel against -30% corrections arising from the term proportional to $A_2 + \Delta_2$ to give 30% corrections to P mass differences.

We show R as a function of $(P_d - P_u)^{(\text{EM})}$ in figs. 2 and 3 for values of \bar{A}_3 corresponding to 0, 30%, and 60% of $P_s - \hat{P}$. We see that if the chiral corrections parameterized by \bar{A}_3 are $\sim 30\%$, and the computed electromagnetic mass differences have errors $\lesssim 30\%$, then

the results for both the D and B systems can comfortably accommodate $R = 44$. This is the value of R predicted by lowest-order chiral perturbation theory for the light meson masses; it is also the value obtained by the $O(m_q^2)$ analysis of ref. [3].

We can use our results to address the question of whether $m_u = 0$ by using a relation between R and m_u/m_d obtained from the light pseudoscalar masses that is valid to $O(m_q^2)$ [5,6] and that predicts $R = 24 \pm 2$ for $m_u = 0$. From figs. 2 and 3, we see that if we assume that \bar{A}_3 is a 30% correction, we require very large corrections in both the D and B electromagnetic mass differences in order to be consistent with $m_u = 0$. If \bar{A}_3 is a 60% correction, the data can accommodate $m_u = 0$.

IV. CONCLUSIONS

We have derived constraints on the light quark mass ratio R from the spectrum of mesons containing a single heavy quark, using the QCD-based computation of the heavy-meson electromagnetic mass differences of ref. [12]. Our results, summarized in figs. 2 and 3, indicate that even when 30% uncertainties are assigned to both the electromagnetic mass differences and the unknown $O(m_q^2)$ chiral corrections, the value of R is bounded away from the value required by $m_u = 0$. While we do not regard this as definitive proof that $m_u \neq 0$, it is striking that the central values of higher-order analyses of both the light pseudoscalar mesons and the heavy mesons prefer $m_u \neq 0$, and large chiral corrections must be invoked in both cases to be consistent with $m_u = 0$.

ACKNOWLEDGEMENTS

We would like to thank J. F. Donoghue, A. Falk, R. F. Lebed, M. Savage and M. Suzuki for helpful discussions. We also thank the Institute for Theoretical Physics at Santa Barbara for hospitality while this work was in progress. MAL is supported in part by DOE contract DE-AC02-76ER03069 and by NSF grant PHY89-04035; RS is supported in part by NSF Grant NSF PHY-92-18167.

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FIGURES

FIG. 1. Diagrams contributing to T in the large- N limit of QCD. The sum over n is over heavy mesons with quantum number $Q\bar{q}$, while the sums over r and s are over light vector mesons with quantum numbers $q\bar{q}$.

FIG. 2. The quark mass ratio R as a function of the spin-independent electromagnetic mass difference in the D system. Our prediction for the electromagnetic mass difference is shown by the vertical dashed line. The bands correspond to the range $0.1 \leq g^2 \leq 0.3$. The chiral corrections parameterized by \bar{A}_3 are assumed to be 0% in the upper band, 30% in the middle band, and 60% in the lower band. The value $R = 44$ is the value obtained from a leading order analysis of the light meson masses, while the value $R = 24$ is the value required by $m_u = 0$.

FIG. 3. The quark mass ratio R as a function of the B electromagnetic mass difference. Our prediction for the electromagnetic mass difference is shown by the vertical dashed line. The bands correspond to the range $0.1 \leq g^2 \leq 0.3$. The chiral corrections parameterized by \bar{A}_3 are assumed to be 0% in the upper band, 30% in the middle band, and 60% in the lower band.

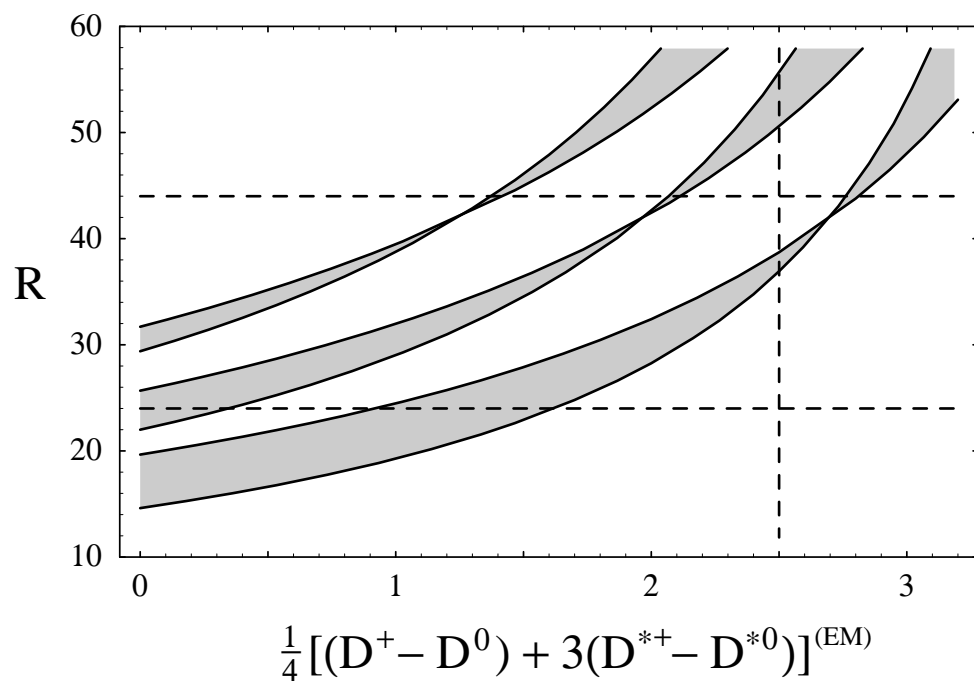


FIG. 2

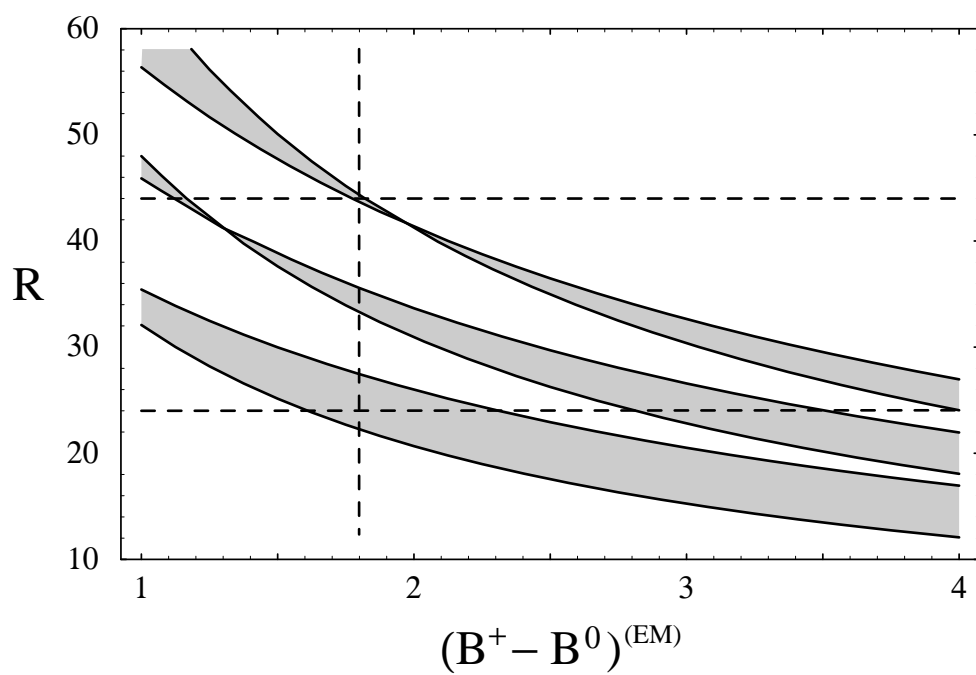


FIG. 3

$$\begin{aligned}
& \text{Diagram 1} = \sum_n \text{Diagram 2}_n + \text{Diagram 3} \\
& + \sum_r \text{Diagram 4}_r + \sum_{r,s} \text{Diagram 5}_{r,s} \\
& + \text{crossed graphs}
\end{aligned}$$

Diagram 1: A horizontal line with a large gray circle in the middle. Two wavy lines extend upwards from the circle.

Diagram 2_n: A horizontal line with two small gray circles. The left circle has a wavy line extending upwards. The right circle has a wavy line extending upwards. A horizontal line segment connects the two circles, with a small 'n' below it.

Diagram 3: A horizontal line with a single black dot. Two wavy lines extend upwards from the dot.

Diagram 4_r: A horizontal line with a black dot. A wavy line extends upwards from the dot. A dashed line segment extends from the dot to the left, ending at a black dot labeled 'r'. A wavy line extends upwards from the dot 'r'.

Diagram 5_{r,s}: A horizontal line with a black dot. A wavy line extends upwards from the dot. Two dashed line segments extend from the dot to the left and right, ending at black dots labeled 'r' and 's' respectively. Wavy lines extend upwards from both dots 'r' and 's'.

(a)

$$\text{Diagram 6} = \text{Diagram 7} + \sum_r \text{Diagram 8}_r$$

Diagram 6: A horizontal line with a large gray circle in the middle. A wavy line extends upwards from the circle.

Diagram 7: A horizontal line with a black dot. A wavy line extends upwards from the dot.

Diagram 8_r: A horizontal line with a black dot. A wavy line extends upwards from the dot. A dashed line segment extends from the dot upwards, ending at a black dot labeled 'r'. A wavy line extends upwards from the dot 'r'.

(b)

FIG. 1